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Existence of positive entire solutions for singular and non-singular quasi-linear elliptic equation ☆

Zuodong Yang

School of Mathematics and Computer Science, Nanjing Normal University, Jiangsu Nanjing 210097, China

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Abstract

We consider the quasilinear elliptic equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u)+\lambda m(x)u^{\gamma}=0$ in domain $\Omega=\mathbf{R}^{\mathbf{N}}$, where $\lambda>0$, $p>1$. Under several hypotheses on the $m(x)$, γ, we obtain the existence of positive entire solutions.

MSC: 35J05; 35J65; 35K05; 35K60

Keywords: Quasi-linear elliptic equation; Positive entire solution; Existence

1. Introduction

In this paper, we are concerned with the existence of positive entire solutions of quasilinear elliptic equations of the type

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u)+\lambda m(x)u^{\gamma}=0,\quad x\in\Omega=\mathbf{R}^{\mathbf{N}}, \tag{1.1}$$

where $m(x)=m(|x|)\in C(\mathbf{R}^{+})$, $\lambda>0$, $p>1$, and $0\geqslant\gamma\geqslant-(p-1)$ or $\gamma>0$. By a positive entire solution of Eq. (1.1), we mean a positive function $u\in C^{1}(\mathbf{R}^{\mathbf{N}})$ which satisfies (1.1) at every point of $\mathbf{R}^{\mathbf{N}}$(see [10] and references therein). If $\lim_{r\to\infty}u(r)=0$, we call it a positive decaying solution.

Equations of the above form are mathematical models occurring in studies of the p-Laplace equation, generalized reaction–diffusion theory [19], non-Newtonian fluid theory [2,25], non-Newtonian filtration [18] and the turbulent flow of a gas in porous medium [8]. In the non-Newtonian fluid theory, the quantity p is characteristic of the medium. Media with $p>2$ are called dilatant fluids and those with $p<2$ are called pseudoplastics. If $p=2$, they are Newtonian fluids.

In recent years, the existence and uniqueness of the positive solutions for the quasilinear eigenvalue problems

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u)+\lambda f(u)=0\quad\text{in }\Omega, \tag{1.2}$$

$$u(x)=0\quad\partial\Omega, \tag{1.3}$$

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E-mail address: zdyang_jin@263.net.

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with $\lambda > 0$, $p > 1$, $\Omega \subset \mathbf{R^N}$, $N \geqslant 2$ have been studied by many authors, see [9–17,27,28,30,32,34] and the references therein. When f is strictly increasing on $\mathbf{R}^+$, $f(0) = 0$, $\lim_{s\to 0^+} f(s)/s^{p-1} = 0$ and $f(s) \leqslant \alpha_1 + \alpha_2 s^{\mu}$, $0 < \mu < p - 1$, $\alpha_1, \alpha_2 > 0$, it was shown in [11] that there exist at least two positive solutions for Eqs. (1.2)–(1.3) when λ is sufficiently large. If $\lim_{s\to 0^+} \inf f(s)/s^{p-1} > 0$, $f(0) = 0$ and the monotonicity hypothesis $(f(s)/s^{p-1})' < 0$ holds for all $s > 0$, it was proved in [15] that problems (1.2)–(1.3) has a unique positive solution when λ is sufficiently large. Moreover, it was also shown in [14] that problems (1.2)–(1.3) have a unique positive large solution and at least one positive small solution when λ is large if f is non-decreasing; there exist $\alpha_1, \alpha_2 > 0$ such that $f(s) \leqslant \alpha_1 + \alpha_2 s^{\beta}$, $0 < \beta < p - 1$; $\lim_{s\to 0^+} f(s)/(s^{p-1}) = 0$, and there exist $T, Y > 0$ with $Y \geqslant T$ such that

$$(f(s)/s^{p-1})' > 0 \quad \text{for } s \in (0, T)$$

and

$$(f(s)/s^{p-1})' < 0 \quad \text{for } s > Y.$$

Recently, Hai [17] considered the case when Ω is an annular domain, and obtained the existence of positive large solutions for problems (1.2)–(1.3) when λ is sufficiently small. Xuan & Chen proved in [3] the singular problem (1.1), (1.3) has a unique positive radial solution if m is a continuous function and positive on $\overline{\Omega} = B_R$ (here B_R is a ball). In contrast to these cases, it seems that very little is known about the existence of entire solutions for (1.1) with singular and non-singular cases except for the work by [10,33,31]. In this paper, we show the existence of positive entire solution to singular problem (1.1) vanishing at infinity under relaxed decay and positivity conditions on the function $m(x)$ with other new condition, our results extend part works by [35,21,7,20], and complement part works by [10,33,31,3]. For $p = 2$, the related results to a singular semilinear elliptic the boundary value problem,

$$\begin{cases} \triangle u + \lambda m(x)u^{\gamma} = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

have been extensively studied when $\Omega \subset \mathbf{R^N}$, see [4–7,20,21,23,24,35]. When $p \neq 2$, the problem becomes more complicated since certain nice properties inherent to the case $p = 2$ seem to be lost or at least difficult to verify. The main differences between $p = 2$ and $p \neq 2$ can be founded in [11,15]. In the last section, we study existence of positive radial entire solutions for non-singular quasilinear elliptic equation, this results extend and complement previous part works by Lair–Shaker [22].

2. Singular case

In this section, we discuss the existence of positive radial entire solutions of problem (1.1) with $\Omega = \mathbf{R^N}$, $N \geqslant 2$. In this case, Eq. (1.1) is equivalent to

$$(r^{N-1}|u'|^{p-2}u')' + \lambda r^{N-1} m(r) u^{\gamma} = 0, \quad r > 0, \quad \lambda > 0, \tag{2.1}$$

where $r = |x|$. If $u = u(r)$ is a solution of (2.1) with $u(0) = \mu$, then u satisfies $u'(0) = 0$. Therefore, Eq. (2.1) is equivalent to

$$u(r) = \mu - \int_0^r \Phi_p^{-1}\left[\lambda s^{1-N} \int_0^s t^{N-1} m(t) u^{\gamma}(t)\,\mathrm{d}t\right] \mathrm{d}s \quad \text{for } r \geqslant 0 \tag{2.2}$$

and if $u = u(r)$ is a positive decaying solution of (2.1), then u satisfies

$$u(r) = \int_r^{\infty} \Phi_p^{-1}\left[\lambda s^{1-N} \int_0^s t^{N-1} m(t) u^{\gamma}(t)\,\mathrm{d}t\right] \mathrm{d}s \quad \text{for } r \geqslant 0, \tag{2.3}$$

where

$$\Phi_p^{-1}(u) = \begin{cases} u^{1/(p-1)} & \text{if } u \geqslant 0, \\ -(-u)^{1/(p-1)} & \text{if } u < 0. \end{cases}$$

Theorem 2.1. *Assume that* $1 < p < N$, $0 \geqslant \gamma > -(p-1)$, *and* $m \in C(\mathbf{R}^+)$, $m > 0$ *for* $r > 0$ *which satisfies*:

(i) *For any* $0 < \varepsilon < (N-p)(p-1-|\gamma|)/(p-1)$,

$$\int_1^\infty r^{p+\varepsilon-1+[(N-p)|\gamma|/(p-1)]} m(r)\,\mathrm{d}r < \infty.$$

(ii) *For* $r \in (0,1)$,

$$m(r) = O(r^{-\delta}), \quad \delta < 1.$$

Then (1.1) *has a positive decaying entire solution for all* $\lambda > 0$.

Proof. We consider Eq. (2.3). Let

$$O = \{u(r) \in \Psi(\mathbf{R}^+) : \theta^{-1}\xi(r) \leqslant u(r) \leqslant \theta\},$$

where θ is a large positive number, $\xi = \min\{1, r^{-(N-p)/(p-1)}\}$, and $\Psi(\mathbf{R}^+)$ is the Frechet space of continuous functions with the topology of uniform convergence on compact intervals.

Let

$$(Lu)(r) = \lambda^{1/(p-1)} \int_r^\infty \left[s^{1-N} \int_0^s t^{N-1} m(t) u^\gamma \,\mathrm{d}t \right]^{1/(p-1)} \mathrm{d}s.$$

Then, for $u \in O$ and $r \geqslant 1$

$$\begin{aligned}(Lu)(r) &\geqslant \lambda^{1/(p-1)} \theta^{\gamma/(p-1)} \int_r^\infty \left[s^{1-N} \int_0^1 t^{N-1} m(t)\,\mathrm{d}t \right]^{1/(p-1)} \mathrm{d}s \\ &\geqslant \lambda^{1/(p-1)} \theta^{\gamma/(p-1)} \left[\frac{p-1}{N-p} \right] \left[\int_0^1 t^{N-1} m(t)\,\mathrm{d}t \right]^{1/(p-1)} r^{-(N-p)/(N-1)}.\end{aligned}$$

When $0 \geqslant \gamma > -(p-1)$ and θ is sufficiently large

$$(Lu)(r) \geqslant \theta^{-1}\xi(r) \quad \text{for fixed } \lambda > 0.$$

For $r < 1$

$$(Lu)(r) \geqslant \lambda^{1/(p-1)} \theta^{\gamma/(p-1)} \int_1^\infty \left[s^{1-N} \int_0^s t^{N-1} m(t)\,\mathrm{d}t \right]^{1/(p-1)} \mathrm{d}s \geqslant \theta^{-1}.$$

On the other hand, for $r \geqslant 1$

$$(Lu)(r) \leqslant \lambda^{1/(p-1)} \theta^{\gamma/(p-1)} \int_r^\infty \left[s^{1-N} \int_0^s t^{N-1} m(t) \xi^\gamma(t)\,\mathrm{d}t \right]^{1/(p-1)} \mathrm{d}s,$$

$$\begin{aligned}(Lu)(r) &\leqslant \lambda^{1/(p-1)} \theta^{\gamma/(p-1)} \int_r^\infty \left[s^{1-N} \left[\int_0^1 + \int_1^s t^{N-1} m(t) \xi^\gamma(t)\,\mathrm{d}t \right] \right]^{1/(p-1)} \mathrm{d}s \\ &\leqslant \lambda^{1/(p-1)} \theta^{\gamma/(p-1)} 2^{1/(p-1)} \int_r^\infty \max \left\{ \left[s^{1-N} \int_0^1 t^{N-1} m(t)\,\mathrm{d}t \right]^{1/(p-1)}, \right. \\ &\qquad \left. \left[s^{1-N} \int_1^s t^{N-1} m(t) \xi^\gamma(t)\,\mathrm{d}t \right]^{1/(p-1)} \right\} \mathrm{d}s \\ &\leqslant \lambda^{1/(p-1)} \theta^{\gamma/(p-1)} 2^{1/(p-1)} (I_1 + I_2).\end{aligned}$$

Here

$$I_1=\int_r^{\infty}\left[s^{1-N}\int_0^1 t^{N-1}m(t)\,\mathrm{d}t\right]^{1/(p-1)}\mathrm{d}s,$$
$$I_2=\int_r^{\infty}\left[s^{1-N}\int_1^s t^{N-1+(N-p)|\gamma|/(p-1)}m(t)\,\mathrm{d}t\right]^{1/(p-1)}\mathrm{d}s.$$

By condition (ii) we know that I_1 is bounded. Now, we discuss I_2

$$\begin{aligned}I_2&=\int_r^{\infty}\left[s^{1-N}\int_1^s t^{N-1+(N-p)|\gamma|/(p-1)}m(t)\,\mathrm{d}t\right]^{1/(p-1)}\mathrm{d}s\\&\leqslant\left[\int_1^{\infty}s^{-1-\varepsilon/(p-1)}\,\mathrm{d}s\right]\left[\int_1^{\infty}t^{N-1+(N-p)|\gamma|/(p-1)}m(t)\,\mathrm{d}t\right]^{1/(p-1)}.\end{aligned}$$

By condition (i) we have that I_2 is bounded. Therefore, there exists $M_3>0$ such that

$$(Lu)(r)\leqslant\lambda^{1/(p-1)}\theta^{\gamma/(p-1)}2^{1/(p-1)}M_3\leqslant\theta,$$

for any fixed $\lambda>0$.

For $r<1$,

$$\begin{aligned}(Lu)(r)&\leqslant\lambda^{1/(p-1)}\theta^{-\gamma/(p-1)}\left[\int_r^1+\int_1^{\infty}\left[s^{1-N}\int_0^s t^{N-1}m(t)\xi^{\gamma}(t)\,\mathrm{d}t\right]^{1/(p-1)}\mathrm{d}s\right]\\&\leqslant\lambda^{1/(p-1)}\theta^{-\gamma/(p-1)}\left[\int_r^1\left[s^{1-N}\int_0^s t^{N-1}m(t)\,\mathrm{d}t\right]^{1/(p-1)}\mathrm{d}s\right]+\lambda^{1/(p-1)}\theta^{-\gamma/(p-1)}M_4.\end{aligned}$$

When $m(r)$ satisfies (i) and (ii), we can directly check that there exists $M_5>0$ such that

$$(Lu)(r)\leqslant\lambda^{1/(p-1)}\theta^{-\gamma/(p-1)}(M_4+M_5)\leqslant\theta\quad\text{for any fixed }\lambda>0.$$

We also know that

$$|(Lu)'(r)|\leqslant\lambda^{1/(p-1)}\theta^{-\gamma/(p-1)}\left[r^{1-N}\int_0^r t^{N-1}m(t)\xi^{\gamma}(t)\,\mathrm{d}t\right]^{1/(p-1)}.$$

Discussing the two cases $r\geqslant 1$ and $r<1$, we find that there exists $M_6>0$ such that

$$|(Lu)'(r)|\leqslant M_6.$$

Using the Schauder–Tychonov fixed point theorem [1,26], we find that a solution in O. □

Remark 1. We are only interested in the case that $\gamma<p-1$. For $\gamma>p-1$ and $m(r)\equiv 1$, we refer to Ref. [10].

From [29], we give the following lemma

Lemma 2.2 (*Weak comparison principle*). *Let Ω be a bounded domain in $\mathbf{R}^{\mathbf{N}}(N\geqslant 2)$ with smooth boundary $\partial\Omega$ and $\theta:(0,\infty)\to(0,\infty)$ is continuous and non-decreasing. Let $u_1,u_2\in W^{1,p}(\Omega)$ satisfy*

$$\int_{\Omega}|\nabla u_1|^{p-2}\nabla u_1\nabla\psi\,\mathrm{d}x+\int_{\Omega}\theta(u_1)\psi\,\mathrm{d}x\leqslant\int_{\Omega}|\nabla u_2|^{p-2}\nabla u_2\nabla\psi\,\mathrm{d}x+\int_{\Omega}\theta(u_2)\psi\,\mathrm{d}x$$

for all non-negative $\psi\in W_0^{1,p}(\Omega)$. Then the inequality

$$u_1\leqslant u_2\quad\text{on }\partial\Omega$$

implies that

$$u_1 \leqslant u_2 \quad \text{in } \Omega.$$

We need to employ a corresponding results by Benji and Zuchi [3] for ball domains.

Lemma 2.3. *Assume that* $m(x) = m(|x|) \in C(0, R)$, $m(r) > 0$ *for all* $r \in (0, R)$, $0 \geqslant \gamma > -(p-1)$, *and* $B_R = \{x \in \mathbf{R^N} : \|x\| < R\}$ *is a N-ball. Then, for all* $\lambda > 0$, *problem*

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda m(x) u^{\gamma} = 0, & x \in B_R, \\ u = 0, & x \in \partial B_R, \end{cases}$$

has at least one positive solution.

Theorem 2.4. *Assume that* $1 < p < N$, $0 \geqslant \gamma > -(p-1)$ *and* $m(x)$ *satisfies*:

(A$_1$) *wherever* $m(x_0) = 0$, $\exists r > 0$ *such that* $m(x) > 0$ *on* $\partial B(x_0, r)$, *where* $B(x_0, r)$ *is the ball of radius r centered at* x_0;
(A$_2$) *for any* $0 < \varepsilon < (N-p)(p-1-|\gamma|)/(p-1)$,

$$\int_1^{\infty} r^{p+\varepsilon-1+[(N-p)|\gamma|/(p-1)]} m(r)\,\mathrm{d}r < \infty;$$

(A$_3$) *for* $r \in (0, 1)$,

$$m(r) = \mathrm{O}(r^{-\delta}), \quad \delta < 1.$$

Then (1.1) *has a positive decaying entire solution for all* $\lambda > 0$.

Proof. We consider

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda m^*(r) u^{\gamma} = 0 \quad \text{for } x \in \mathbf{R^N} \text{ and } \lambda > 0, \tag{2.4}$$

where $m^*(x) = m(x) + \psi(|x|)/k$, $k = 1, 2, \ldots$, ψ is any smooth function for which $\psi(t) > 0$, for $t \geqslant 0$ and ψ satisfies (A$_2$)–(A$_3$). We have $m^* > 0$ for all $x \in \mathbf{R^N}$. By Theorem 2.1, (2.4) has a unique positive solution $u_k \to 0$ as $|x| \to \infty$. Clearly

$$\operatorname{div}(|\nabla u_{k+1}|^{p-2}\nabla u_{k+1}) + \lambda\left(m(x) + \frac{\psi(|x|)}{k}\right) u_{k+1}^{\gamma} \geqslant 0.$$

From Lemma 2.2, it follows that $u_k \geqslant u_{k+1}$. Thus we have a decreasing sequence

$$u_1 \geqslant u_2 \geqslant \cdots \geqslant u_{k+1} \geqslant \cdots > 0.$$

Let $u(x)$ be the point-wise limit function of the sequence $\{u_k\}_1^{\infty}$. We have $u_k \in C^{1+\alpha}$ for all $k \in \mathbf{N}$ and $u(x) \geqslant 0$ for all $x \in \mathbf{R^N}$.

We now prove that $u \in C^{1+\alpha}_{\mathrm{loc}}(\mathbf{R^N})$ and is consequently a solution of Eq. (1.1). Note that now our solution sequence $\{u_k\}_1^{\infty}$ is monotone decreasing and thus does not have a uniform positive lower bound. However, we need to prove that our solution sequence $\{u_k\}_1^{\infty}$ have a positive lower bound for any ball in $\mathbf{R^N}$. To achieve this, we need to consider only two cases concerning the neighborhood of an arbitrarily chosen point x_0: (i) $m(x_0) = 0$; (ii) $m(x_0) > 0$. In what follows we show that in either case we can find a ball centered at x_0 such that the sequence $\{u_k\}$ is uniformly bounded.

Suppose $m(x_0) = 0$. By (A_1), there exists a ball $B(x_0, r)$ such that $m(x) > 0$ on $\partial B(x_0, r)$. Let $\overline{x}$ be any point on $\partial B(x_0, r)$ and $B(\overline{x}, \overline{r})$ be a ball centered at $\overline{x}$ of radius $\overline{r}$ that does not contain any of the zeros of $m(x)$. By Lemma 2.3, (1.1) has a unique positive solution, which we call $\overline{u}$, in $\Omega = B(\overline{x}, \overline{r})$ vanishing on the boundary $\partial\Omega$. We have

$$\operatorname{div}(|\nabla \overline{u}|^{p-2}\nabla \overline{u}) + \lambda[m(x) + \psi(|x|)/k]\overline{u}^{\gamma} \geqslant 0.$$

That is, $\overline{u}$ is a lower solution of (2.4). Application the weak comparison principle yields that $u_k \geqslant \overline{u}$ for all k. Write $\overline{\varepsilon} = \min_{B(\overline{x}, r/2)} \overline{u} > 0$. Since $\partial\Omega$ is bounded, it can be covered by finitely many such balls. Let ε be the minimum of all such $\overline{\varepsilon}$. Then we have $u_k \geqslant \varepsilon$ for all k along the boundary $\partial\Omega$. We claim that this is in fact true for all $x \in \overline{\Omega}$. In fact,

$$-\mathrm{div}(|\nabla u_k|^{p-2}\nabla u_k) > 0 = -\mathrm{div}(|\nabla \varepsilon|^{p-2}\nabla \varepsilon),$$

from Lemma 2.2, we have $u_k \geqslant \varepsilon$ for all $x \in \overline{\Omega}$. Thus we have a uniform lower bound ε for $\{u_k\}_1^\infty$ in $\overline{\Omega}$.

Suppose $m(x_0) > 0$. Since $m(x)$ is continuous, we can find a ball of some radius, say $r > 0$, centered at x_0 such that $m(x) > 0$ for all $x \in \overline{B(x_0, r)}$. By Lemma 2.3, (1.1) has a unique positive solution, say u_0, in $B(x_0, r)$ that vanishes on $\partial B(x_0, r)$. Clearly u_0 is a lower solution of (2.4). Lemma 2.2 again implies that $u_k \geqslant u_0$ for all k. We can assume that $u_k \geqslant u_0 \geqslant \varepsilon_0$ in $B(x_0, r_0/2)$ for some ε_0. Since x_0 was arbitrarily chosen, we have shown that for any $x \in \mathbf{R^N}$, there exists a ball of some positive radius centered at this point where the sequence $\{u_k\}_1^\infty$ has a uniform positive lower bound. By the analysis given above, we have that $u(x) \in C^{1+\alpha}_{\mathrm{loc}}(\mathbf{R^N})$ and $u(x)$ is a positive solution of (1.1). $\square$

3. Non-singular case

For $0 < \gamma \leqslant p - 1$, we obtain the following main theorem:

Theorem 3.1. *Assume that* $0 < \gamma \leqslant p - 1$, $m(x) = m(|x|) \in C(\mathbf{R^N})$ *is non-negative and non-trivial, and*

$$\int_0^\infty (r\psi(r))^{1/(p-1)}\,\mathrm{d}r < \infty, \tag{3.1}$$

where $\psi(r) = \max_{|x| \leqslant r} m(x)$. *Then the equation*

$$\mathrm{div}(|\nabla u|^{p-2}\nabla u) = m(|x|)u^\gamma, \quad x \in \mathbf{R^N} \tag{3.2}$$

has a positive radial solution in $\mathbf{R^N}$.

Proof. The proof is similar to [22,29], so we omit the details. $\square$

Remark 2. If condition (3.1) of Theorem 3.1 is replaced by

$$\int_0^\infty (r\rho(r))^{1/(p-1)}\,\mathrm{d}r = \infty,$$

where $\rho(r) = \min_{|x| \leqslant r} m(x)$, then (3.2) has an explosive positive radial solution in $\mathbf{R^N}$. In fact, we prove that $u(r) \to \infty$ as $r \to \infty$. This can be done by establishing the following relation:

$$\int_0^\infty (r\rho(r))^{1/(p-1)}\,\mathrm{d}r = \infty,$$

which implies that

$$\int_0^\infty s^{(1-N)/(p-1)} \left(\int_0^s t^{N-1} m(t)\,\mathrm{d}t \right)^{1/(p-1)} \mathrm{d}s = \infty. \tag{3.3}$$

Indeed, note that

$$\int_0^r s^{(1-N)/(p-1)} \left(\int_0^s t^{N-1} m(t)\,\mathrm{d}t \right)^{1/(p-1)} \mathrm{d}s$$
$$\geqslant \int_0^s s^{(1-N)/(p-1)} (\rho(s))^{1/(p-1)} \left(\int_0^s t^{N-1}\,\mathrm{d}t \right)^{1/(p-1)} \mathrm{d}s = 1/N \int_0^r (s\rho(s))^{1/(p-1)}\,\mathrm{d}s.$$

Letting $r \to \infty$, we get Eq. (3.3).

For $\gamma > 0$, we give some new results which are as follows:

Theorem 3.2. *Assume that* $1 < p < N$, $\gamma > 0$, *and* $m \in C^1(\mathbf{R}, \mathbf{R}^+)$ *satisfies*

$$\frac{\mathrm{d}}{\mathrm{dt}}[t^{[(N+1)p-N-\gamma(N-p)]/p}m(t)] \leqslant 0 \quad \textit{for } t > 0. \tag{3.4}$$

Then for any $\lambda > 0$ *and* $\alpha > 0$, *equation*

$$\mathrm{div}(|Du|^{p-2}Du) + \lambda m(|x|)u^{\gamma} = 0, \quad x \in \mathbf{R^N} \tag{3.5}$$

has a positive radial entire solution u *such that* $u(0) = \alpha$.

Remark 3. If condition $m(x) = m(|x|) > 0$ of Theorem 3.2 is replaced by

Wherever $m(x_0) = 0$, $\exists r > 0$ such that $m(x) > 0$ on $\partial B(x_0, r)$, where $B(x_0, r)$ is a ball of radius r centered at x_0.
Then conclusion of Theorem 3.2 still holds.
To prove Theorem 3.2, we give the following lemmas

Lemma 3.3. *Assume that* $u = u(r)$ *is a radial solution of* (3.5) *and there exists* $r_0 > 0$ *such that* $u > 0$ *in* $(0, r_0)$ *and* $u' < 0$ *in* $(0, r_0)$, $u(r_0) = 0$, *then* $u'(r_0) \neq 0$.

Proof. By (3.5) we have that u satisfies

$$(r^{N-1}|u'|^{p-2}u')' + \lambda r^{N-1}m(r)u^{\gamma} = 0, \quad 0 < r < r_0. \tag{3.6}$$

Suppose $u'(r_0) = 0$. Integrating (3.6) on $(0, r_0)$ we obtain

$$r^{N-1}|u'(r)|^{p-2}u'(r) = \lambda \int_r^{r_0} s^{N-1}m(s)u^{\gamma}(s)\,\mathrm{d}s.$$

This is a contradiction of $u > 0$ and $u' < 0$ in $(0, r_0)$. $\square$

Lemma 3.4. *Assume that* (3.4) *holds*, $u \in C^1(0, \infty)$ *is a radial solution of Eq.* (3.5) *with* $u(0) = \alpha > 0$. *Then* $u > 0$ *in* $(0, \infty)$.

Proof. Suppose that there exists a $r_0 > 0$ such that $u > 0$ in $(0, r_0)$ and $u(r_0) = 0$. Integration of (3.5) in $(0, r)$ shows that $u' < 0$ in $(0, r_0)$. Let

$$V(r) = r^{N-1}|u'(r)|^{p-2}u'(r) + \frac{p-1}{N-p}r^N|u'(r)|^p + \frac{p\lambda}{(N-p)(\gamma+1)}r^N m(r)u^{\gamma+1},$$

then

$$V'(r) = \frac{p\lambda}{(N-p)(\gamma+1)}r^{(1+\gamma)(N-p)/p}[r^{[(N+1)p-N-\gamma(N-p)]/p}m(r)]'u^{\gamma+1}.$$

It follows from (3.4) that $V'(r) \leqslant 0$ in $(0, r_0)$. Therefore, $V(r_0) \leqslant V(0) = 0$. This implies that $u'(r_0) = 0$ which contradicts Lemma 3.3. $\square$

Proof of Theorem 3.2. Consider the solutions to the following Cauchy problem:

$$(r^{N-1}|u'|^{p-2}u') + \lambda r^{N-1}m(r)u^{\gamma} = 0, \quad 0 < r < \infty,$$

$$u(0) = \alpha, \ u'(0) = 0.$$

It is well-known that there exists a solution $u \in C^1(0, \infty)$ such that

$$u(r) = \alpha - \int_0^r \Phi_p^{-1}\left[\lambda s^{1-N} \int_0^s t^{N-1}m(t)u^{\gamma}(t)\,\mathrm{d}t\right]\mathrm{d}s \quad \text{for } r \geqslant 0.$$

By Lemma 3.4, we know $u(r) > 0$ for $r > 0$.

We note that equation

$$(r^{N-1}|u'|^{p-2}u') + \lambda r^{N-1} f(r,u) = 0, \quad 0 < r < \infty \tag{3.7}$$

has no bounded positive radial entire solution for $p \geqslant N$ and $0 \not\equiv f \geqslant 0$. This is well-known and easily proved in the case $p=2$. Indeed, if such a solution exists, $r^{N-1}\Phi_p(u_r)$ would be non-increasing in $[0,\infty)$ by (3.7), here $\Phi_p(u) = |u|^{p-2}u$. Hence, $\Phi_p(u_r(r_1)) < 0$ for some $r_1 > 0$. Since $r^{N-1}\Phi_p(u_r(r)) \leqslant -k$ for $r > r_1$, where $k = -r_1^{N-1}\Phi_p(u_r(r_1)) > 0$. Then

$$u_r \leqslant -\Phi_p^{-1}(kr^{1-N}) = -k^{1/(p-1)} r^{(1-N)/(p-1)} \quad \text{for } r > r_1.$$

Using $p \geqslant N$ and integrating over (r_1, r), we get the contradiction $u(r) < 0$. Now, we give some new existence results for $f \leqslant 0$. $\square$

Theorem 3.5. *Assume that* $f : \overline{\mathbf{R}}^+ \times \mathbf{R}^+ \to \mathbf{R}^-$ *is continuous,* $|f(r,\cdot)|$ *is non-decreasing for all* $r \geqslant 0$ *and*

$$\begin{cases} \int_e^\infty r^{N-1}|f(r, 3\theta \ln r)|\,\mathrm{d}r < \infty & \text{if } p = N, \\ \int_1^\infty r^{N-1}|f(r, 3\theta r^{(p-N)/(p-1)})|\,\mathrm{d}r < \infty & \text{if } p > N, \end{cases}$$

for some large positive constant $\theta > 0$. *Then there exists a constant* $\overline{\lambda} > 0$ *such that* $0 < \lambda \leqslant \overline{\lambda}$, *the equation*

$$\operatorname{div}(|Du|^{p-2}Du) + \lambda f(|x|, u) = 0 \quad \text{for } |x| = r \geqslant 0 \tag{3.8}$$

has at least one positive entire solution satisfying

$$\begin{cases} \theta \leqslant u(|x|) \leqslant 3\theta \ln |x| & \text{for } |x| = r \geqslant e, \ \ p = N, \\ \theta \leqslant u(|x|) \leqslant 3\theta r^{(p-N)/(p-1)} & \text{for } |x| = r \geqslant 1, \ \ p > N. \end{cases}$$

Proof. We consider Eq.(2.2). Define

$$O = \{u \in \Psi(\overline{\mathbf{R}}^+) : \theta \leqslant u(r) \leqslant 3\theta\rho(r)\},$$

where $\Psi(\overline{\mathbf{R}}^+)$ is as in Theorem 2.1,

$$\rho(r) = \begin{cases} \max\{1, \ln r\} & \text{for } p = N, \\ \max\{1, r^{(p-N)/(p-1)}\} & \text{for } p > N. \end{cases}$$

Define

$$(Lu)(r) = \theta + \int_0^r \Phi_p^{-1}\left[\lambda s^{1-N} \int_0^s t^{N-1}|f(t, u(t))|\,\mathrm{d}t\right] \mathrm{d}s.$$

Thus

$$(Lu)(r) \geqslant \theta \quad \text{for all } u \in O.$$

For $r \geqslant e$, $p = N$(for $r \geqslant 1$, $p > N$ being similar) and $u \in O$

$$
\begin{aligned}
(Lu)(r) &\leqslant \theta + \left[\left(\int_0^e + \int_e^r\right) \lambda^{1/(p-1)} s^{(1-N)/(p-1)} \left[\int_0^s t^{N-1} |f(t, u(t))| \,\mathrm{d}t\right]^{1/(p-1)} \mathrm{d}s\right] \\
&\leqslant \theta + \left[\int_0^e \lambda^{1/(p-1)} s^{(1-N)/(p-1)} \left[\int_0^s t^{N-1} |f(t, 3\theta)| \,\mathrm{d}t\right]^{1/(p-1)} \mathrm{d}s\right. \\
&\quad + \int_e^r \lambda^{1/(p-1)} s^{(1-N)/(p-1)} \left\{\left[\int_0^e t^{N-1} |f(t, 3\theta)| \,\mathrm{d}t\right]^{1/(p-1)} \mathrm{d}s\right. \\
&\quad \left.\left. + \left[\int_e^\infty t^{N-1} |f(t, 3\theta \ln t)| \,\mathrm{d}t\right]^{1/(p-1)} \mathrm{d}s\right\}\right] \\
&\leqslant \theta + \lambda^{1/(p-1)} [M_7 + M_8 \ln r]
\end{aligned}
$$

for $0 < r < e$, we can easily prove that

$$(Lu)(r) \leqslant \theta + \lambda^{1/(p-1)} M_9.$$

Here M_7, M_8, M_9 are positive numbers. Therefore, we can choose θ and $\overline{\lambda} > 0$ such that for $0 < \lambda \leqslant \overline{\lambda}$, it follows that

$$(Lu)(r) \leqslant 3\theta\rho(r),$$

the remainder of the proof is the same as that of Theorem 2.1. □

Remark 4. For $1 < p < N$, there are same results as Theorem 3.5.

References

[1] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach space, SIAM Rev. 18 (1976) 620–709.

[2] G. Astrita, G. Marrucci, Principles of Non-Newtonian Fuid Mechanics, McGraw-Hill, New York, 1974.

[3] X. Benjin, Z. Chen, Solvability of singular quasilinear elliptic equation, Chinese Ann. Math. 20A (1) (1999) 117–128.

[4] H. Brezis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, Comm. Pure Appl. Math. 36 (1983) 437–477.

[5] M.G. Crandall, P.H. Rabinowitz, Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems, Arch. Rational Mech. Anal. 58 (1975) 207–218.

[6] J.I. Diaz, J.M. Morel, L. Oswald, An elliptic equation with singular nonlinearity, Comm. Partial Differential Equations 12 (12) (1987) 1333–1344.

[7] A. Edelson, Entire solutions of singular elliptic equations, J. Math. Anal. Appl. 139 (1989) 523–532.

[8] J.R. Esteban, J.L. Vazquez, On the equation of turbulent filtration in one-dimensional porous media, Nonlinear Anal. 10 (1982) 1303–1325.

[9] I.M. Gelfand, Some problems in the theory of quasilinear equations, Amer. Math. Soc. Transla. 2 (29) (1963) 295–381.

[10] M. Guedda, L. Veron, Local and global properties of solutions of quasilinear elliptic equations, J. Differential Equations 76 (1988) 159–189.

[11] Z. Guo, Some existence and multiplicity results for a class of quasilinear elliptic eigenvalue problems, Nonlinear Anal. 18 (1992) 957–971.

[12] Z. Guo, Existence and uniqueness of positive radial solutions for a class of quasilinear elliptic equations, Appl. Anal. 47 (1992) 173–190.

[13] Z. Guo, Boundary value problems of a class of quasilinear ordinary differential equations, Differential Integral Equations 6 (3) (1993) 705–719.

[14] Z. Guo, On the number of positive solutions for quasilinear elliptic eigenvalue problems, Nonlinear Anal. 27 (2) (1996) 229–247.

[15] Z. Guo, J.R.L. Webb, Uniqueness of positive solutions for quasilinear elliptic equations when a parameter is large, Proc. Roy. Soc. Edinburgh 124A (1994) 189–198.

[16] Z. Guo, Z. Yang, Some uniqueness results for a class of quasilinear elliptic eigenvalue problems, Acta Math. Sinica (N.S.) 14 (2) (1998) 245–260.

[17] D.D. Hai, Positive solutions of quasilinear boundary value problems, J. Math. Anal. Appl. 217 (1998) 672–686.

[18] A.S. Kalashnikov, On a nonlinear equation appearing in the theory of non-stationary filtration, Trudy Sem. Petrovsk. 5 (1978) 60–68 (in Russian).

[19] H.B. Keller, D.S. Cohen, Some positone problems suggested by nonlinear heat generation, J. Math. Mech. 16 (1967) 1361–1376.

[20] T. Kusano, C.A. Swanson, Entire positive solutions of singular semilinear elliptic equations, Japan J. Math. 11 (1985) 145–155.

[21] A.V. Lair, A.W. Shaker, Entire solution of a singular semilinear elliptic problem, J. Math. Anal. Appl. 200 (1996) 498–505.

[22] A.V. Lair, A.W. Wood, Large solutions of sublinear elliptic equations, Nonlinear Anal. 39 (2000) 745–753.

[23] A.C. Lazer, P.J. McKenna, On a singular nonlinear elliptic boundary value problem, Proc. Amer. Math. Soc. 111 (3) (1991) 721–730.

[24] P.L. Lions, On the existence of positive solutions of semilinear elliptic equations, SIAM Rev. 24 (1982) 441–467.
[25] L.K. Martinson, K.B. Pavlov, Unsteady shear flows of a conducting fluid with a rheological power law, Magnit. Gidrodinamika 2 (1971) 50–58.
[26] D.S. Mitrinovic, Analytic Inequalities, Springer, Berlin, New York, 1970.
[27] W.M. Ni, J. Serrin, Nonexistence theorems for singular solutions of quasilinear partial differential equations, Comm. Pure Appl. Math. 39 (1986) 379–399.
[28] M.D. Pino, M. Elgueta, R. Manasevich, A homotopic deformation along p of a Laray–Schauder degree result and existence for $(|u'|^{p-2}u')' + f(t,u) = 0$, $u(0) = u(T) = 0$, $p > 1$, J. Differential Equations 80 (1989) 1–13.
[29] L. Qishao, Z. Yang, E.H. Twizell, Existence of entire explosive positive solutions of quasilinear elliptic equations, Appl. Math. Comput. 148 (2004) 359–372.
[30] S. Sakaguchi, Concavity properties of solutions to some degenerate quasilinear elliptic Dirichlet problems, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (IV) 14 (1987) 403–421.
[31] Z. Yang, Existence of positive bounded entire solutions for quasilinear elliptic equations, Appl. Math. Comput. 156 (2004) 743–754.
[32] Z. Yang, Z. Guo, On the structure of positive solutions for quasilinear ordinary differential equations, Appl. Anal. 58 (1995) 31–51.
[33] Z. Yang, Q. Lu, Existence and multiplicity of positive entire solutions for a class of quasilinear elliptic equation, J. Beijing Univ. Aeronaut. Astronaut. 27 (2001) 217–220.
[34] Z. Yang, H. Yang, A priori for a quasilinear elliptic P.D.E non-positone problems, Nonlinear Anal. 43 (2) (2001) 173–181.
[35] Z. Zhang, On a Dirichlet problem with singular nonlinearity, J. Math. Anal. Appl. 194 (1) (1995) 103–113.